1 UCSV model of Stock and Watson (2007)

The unobserved component model with stochastic volatility can be defined as:

$$y_t = \tau_t + \epsilon_t, \epsilon_t \sim N(0, e^{h_t}), \tag{1}$$

$$\tau_t = \tau_{t-1} + \zeta_t, \zeta_t \sim N(0, e^{g_t}), \tag{2}$$

$$h_t = h_{t-1} + v_t, v_t \sim N(0, \sigma_h^2), \tag{3}$$

$$g_t = g_{t-1} + \xi_t, \xi_t \sim N(0, \sigma_g^2), \tag{4}$$

Here y_t can be defined as CPI inflation or Real GDP growth. We denote the initial conditions and priors: $\tau_0 \sim N(\bar{\tau}_0, V_\tau)$, $h_1 \sim N(0, V_h)$, $g_1 \sim N(0, V_g)$, $\sigma_h^2 \sim IG(\nu_1, S_1)$ and $\sigma_g^2 \sim IG(\nu_2, S_2)$. We can stack the above equations over time T

$$\mathbf{y} = \tau + \epsilon, \epsilon \sim N(0, \Sigma), \tag{5}$$

$$\mathbf{H}\tau = \alpha + \zeta, \zeta \sim N(0, \Omega), \tag{6}$$

$$\mathbf{H}\mathbf{h} = v, v \sim N(0, \mathit{\Lambda}_1), \tag{7}$$

$$\mathbf{H}\mathbf{g} = \xi, \xi \sim N(0, \mathit{\Lambda}_2), \tag{8}$$

where $\mathbf{y} = (y_1, \ldots, y_T)'$, $\tau = (\tau_1, \ldots, \tau_T)'$, $\mathbf{h} = (h_1, \ldots h_T)'$, $\mathbf{g} = (g_1, \ldots g_T)'$, $\alpha = (\tau_0, 0, \ldots, 0)'$, $\Sigma = \text{diag}(e^{h_1}, \ldots, e^{h_T})$, $\Omega = \text{diag}(e^{g_1}, \ldots, e^{g_T})$, $\mathit{\Lambda}_1 = \text{diag}(V_h, \sigma_h^2, \ldots, \sigma_h^2)$, $\mathit{\Lambda}_2 = \text{diag}(V_g, \sigma_g^2, \ldots, \sigma_g^2)$ and

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

From equation (5) and (6) we can derive the conditional posterior for τ

$$(\tau|\mathbf{y}, \mathbf{h}, \mathbf{g}, \tau_0, \sigma_h^2, \sigma_g^2) \propto (\mathbf{y}|\tau, \mathbf{h})(\tau|\mathbf{g}, \tau_0)$$

Note $(\tau|\mathbf{g}, \tau_0) \sim N(\bar{\alpha}, (\mathbf{H}'\Omega^{-1}\mathbf{H})^{-1})$ where $\bar{\alpha} = \mathbf{H}^{-1}\alpha$ and thus

$$(\tau|\mathbf{y}, \mathbf{h}, \mathbf{g}, \tau_0, \sigma_h^2, \sigma_g^2) \propto \exp -\frac{1}{2}[(\mathbf{y} - \tau)'\Sigma^{-1}(\mathbf{y} - \tau)]\exp -\frac{1}{2}[(\tau - \bar{\alpha})'(\mathbf{H}'\Omega^{-1}\mathbf{H})(\tau - \bar{\alpha})],$$

$$(\tau|\mathbf{y},\mathbf{h},\mathbf{g},\tau_0,\sigma_h^2,\sigma_g^2) \propto \exp -\frac{1}{2}[\tau'(\Sigma^{-1} + (\mathbf{H}'\Omega^{-1}\mathbf{H}))\tau - 2\tau'(\Sigma^{-1}\mathbf{y} + (\mathbf{H}'\Omega^{-1}\mathbf{H})\bar{\alpha})],$$

Thus

$$(\tau|\mathbf{y},\mathbf{h},\mathbf{g},\tau_0,\sigma_h^2,\sigma_g^2) \sim N(\hat{\tau},\mathbf{D}_\tau),$$

where $\mathbf{D}_\tau = (\Sigma^{-1} + (\mathbf{H}'\Omega^{-1}\mathbf{H}))^{-1}$ and $\hat{\tau} = \mathbf{D}_\tau(\Sigma^{-1}\mathbf{y} + (\mathbf{H}'\Omega^{-1}\mathbf{H})\bar{\alpha}))$. We can use same algorithm discussed in lab 2 notes to draw from this multivariate normal distribution. Next, the conditional posterior for τ_0 is

$$(\tau_0|\mathbf{y},\mathbf{h},\mathbf{g},\sigma_h^2,\sigma_g^2) \sim N(\hat{\tau}_0,K_{\tau_0}),$$

where $K_{\tau_0} = (\frac{1}{V_\tau} + \frac{1}{e^{g_1}})^{-1}$ and $\hat{\tau}_0 = K_{\tau_0}(\frac{\tau_1}{e^{g_1}} + \frac{\bar{\tau}_0}{V_\tau})$. To draw $\mathbf{h}$ and $\mathbf{g}$, it is a bit more difficult and we need to use an auxiliary mixture sampler of Kim, Shepherd and Chib (1998). For more information, I suggest you read Chapter 7 of Joshua Chan Bayesian Macroeconometrics notes. Lastly, the conditional posteriors for σ_h^2 and σ_g^2 are

$$(\sigma_h^2|\mathbf{y},\mathbf{h},\mathbf{g},\tau,\tau_0,\sigma_g^2) \sim IG(\nu_1 + \frac{T-1}{2}, S_1 + \frac{\sum_{t=2}^T (h_t - h_{t-1})^2}{2}),$$

$$(\sigma_g^2|\mathbf{y},\mathbf{h},\mathbf{g},\tau,\tau_0,\sigma_h^2) \sim IG(\nu_2 + \frac{T-1}{2}, S_2 + \frac{\sum_{t=2}^T (g_t - g_{t-1})^2}{2}).$$

2 Extension to the UCSV

We may want to extend the UCSV model to add an exogenous variable, such as the Unemployment rate, in the measurement equation of (1)

$$y_t = \tau_t + x_t\beta + \epsilon_t, \epsilon_t \sim N(0, e^{h_t}),$$

where x_t is the unemployment rate and stack it over time $\mathbf{x} = (x_1,\ldots,x_T)'$. We assume the same priors as in section 1 and we have an additional prior on $\beta \sim N(\beta_0, V_\beta)$. The conditional posterior for τ is slight modified to be

$$(\tau|\mathbf{y},\mathbf{h},\mathbf{g},\tau_0,\sigma_h^2,\sigma_g^2,\beta) \sim N(\hat{\tau},\mathbf{D}_\tau),$$

where $\mathbf{D}_\tau = (\Sigma^{-1} + (\mathbf{H}'\Omega^{-1}\mathbf{H}))^{-1}$ and $\hat{\tau} = \mathbf{D}_\tau(\Sigma^{-1}(\mathbf{y} - \mathbf{x}\beta) + (\mathbf{H}'\Omega^{-1}\mathbf{H})\bar{\alpha}))$. Next, the conditional posterior of β is

$$(\beta|\mathbf{y},\tau,\mathbf{h},\mathbf{g},\tau_0,\sigma_h^2,\sigma_g^2) \sim N(\hat{\beta},K_\beta),$$

where $K_\beta = (V_\beta^{-1} + \mathbf{x}'\Sigma^{-1}\mathbf{x})^{-1}$ and $\hat{\beta} = K_\beta(\mathbf{x}'\Sigma^{-1}(\mathbf{y} - \tau) + V_\beta^{-1}\beta_0)$. All the other conditional posteriors are basically the same as before.