1 Time-varying parameter AR model

We can define a time-varying parameter (TVP) AR(p) model as

$$y_t = c_t + \rho_{1,t} y_{t-1} + \ldots + \rho_{p,t} y_{t-p} + \epsilon_t, \epsilon_t \sim N(0, \sigma^2), \tag{1}$$

and we can rewrite the above equation in linear regression matrix form

$$y_t = \mathbf{x}_t \beta_t + \epsilon_t, \epsilon_t \sim N(0, \sigma^2), \tag{2}$$

The TVP follows a randow walk assumption

$$\beta_t = \beta_{t-1} + \eta_t, \eta_t \sim N(0, \Omega), \tag{3}$$

where

$$\mathbf{x}_t = \begin{bmatrix} 1 & y_{t-1} & y_{t-2} & \ldots & y_{t-p} \end{bmatrix}, \beta_t = \begin{bmatrix} c_t \\ \rho_{1,t} \\ \rho_{2,t} \\ \vdots \\ \rho_{p,t} \end{bmatrix}_{k \times 1},$$

Next, we can stack (2) and (3) over time T

$$\mathbf{y} = \mathbf{X}\beta + \epsilon, \epsilon \sim N(0, \Sigma), \tag{4}$$

where $\mathbf{y} = (y_{p+1}, \ldots, y_T)'$, $\mathbf{X} = \mathrm{diag}(\mathbf{x}_1, \ldots, \mathbf{x}_T)'$, $\beta = (\beta_1, \ldots, \beta_T)'$ is a $Tk \times 1$ vector, $\Omega = \mathrm{diag}(\omega_1^2, \ldots, \omega_k^2)$, and $\Sigma = \mathrm{diag}(\sigma^2, \ldots, \sigma^2)$. If we assume the intial condition for $\beta_0 \sim N(0, V_\beta)$, then we can stack (3) over T

$$\mathbf{H}\beta = \eta, \eta \sim N(0, \mathbf{S}), \tag{5}$$

where $\eta = (\eta_1, \ldots, \eta_T)'$, $\mathbf{S} = \mathrm{diag}(V_\beta, \Omega, \ldots, \Omega)$, and

$$\mathbf{H} = \begin{bmatrix} \mathbf{I}_k & 0 & & \cdots & 0 \\ -\mathbf{I}_k & \mathbf{I}_k & 0 & & 0 \\ 0 & -\mathbf{I}_k & \mathbf{I}_k & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\mathbf{I}_k & \mathbf{I}_k \end{bmatrix},$$

By the change of variables, equation (5) becomes

$$\beta \sim N(0, (\mathbf{H}'\mathbf{S}^{-1}\mathbf{H})^{-1}). \tag{6}$$

Finally, to complete the model, we assume priors for

$$\sigma^2 \sim IG(\nu_1, S_1), \tag{7}$$

$$\omega_i^2 \sim IG(\nu_2, S_2). \tag{8}$$

Here IG and N are denoted as the inverse-gamma distribution and the normal distribution respectively.

1.1 Draw β_t

To derive the conditional posterior of β_t, we use (4), (which is the likelihood) and (6),

$$(\beta|\mathbf{y}, \sigma^2, \Omega) \propto p(\mathbf{y}|\beta, \sigma^2, \omega^2)p(\beta),$$

$$\propto \exp[-\frac{1}{2}(\mathbf{y} - \mathbf{X}\beta)'\Sigma^{-1}(\mathbf{y} - \mathbf{X}\beta)]\exp[-\frac{1}{2}\beta'(\mathbf{H}'\mathbf{S}^{-1}\mathbf{H})\beta],$$

$$\propto \exp[-\frac{1}{2}(\beta'(\mathbf{X}'\Sigma^{-1}\mathbf{X} + \mathbf{H}'\mathbf{S}^{-1}\mathbf{H})\beta - 2\beta'\mathbf{X}'\Sigma^{-1}\mathbf{y})],$$

Thus, the conditional posterior for β_t is

$$(\beta|\mathbf{y}, \sigma^2, \Omega) \sim N(\hat{\beta}, \mathbf{K}_\beta),$$

where

$$\mathbf{K}_\beta = (\mathbf{X}'\Sigma^{-1}\mathbf{X} + \mathbf{H}'\mathbf{S}^{-1}\mathbf{H})^{-1}, \hat{\beta} = \mathbf{K}_\beta(\mathbf{X}'\Sigma^{-1}\mathbf{y}).$$

Since the precision matrix $\mathbf{K}_\beta$ is a band matrix, one can sample from $(\beta|\mathbf{y}, \sigma^2, \Omega)$ efficiently using the algorithm in Chan and Jeliazkov (2009).

1.2 Draw σ^2 and ω^2

The conditional posteriors of these variances are standard and straightforward to draw

$$(\sigma^2|\mathbf{y}, \beta, \Omega) \sim IG(\nu_1 + \frac{T}{2}, S_1 + \frac{1}{2}\sum_{t=1}^{T}(y_t - \mathbf{X}_t\beta_t)^2), \tag{9}$$

$$(\omega_i^2|\mathbf{y}, \beta, \sigma^2) \sim IG(\nu_2 + \frac{T-1}{2}, S_2 + \frac{1}{2}\sum_{t=2}^{T}(\rho_{i,t} - \rho_{i,t-1})^2), \text{for } i = 1, \ldots, k. \tag{10}$$

References

[1] Chan, J. C., & Jeliazkov, I. (2009). Efficient simulation and integrated likelihood estimation in state space models. International Journal of Mathematical Modelling and Numerical Optimisation, 1(1-2), 101-120.